A Simplified PML for Use with the FDTD Method

Dennis M. Sullivan, *Senior Member, IEEE*

***Abstract*—A recent advance in the use of the finite-difference time-domain (FDTD) method has been the introduction of the perfectly matched layer (PML) to act as the absorbing boundary condition. This letter suggests using fictitious magnetic and electric displacement (as opposed to electric field) conductivities in order to better isolate the PML from the rest of the FDTD problem. It further describes the implementation for the PML directly from the FDTD formulation itself. This results in an analysis that is much easier to understand and to program.**

I. Introduction

THE finite-difference time-domain (FDTD) method [1] has become one of the most widely used methods of electromagnetic simulation [2], [3]. A key issue in the implementation of the FDTD method, regardless of its application, has always been the termination of the problem space via absorbing boundary conditions (ABC). Early approaches included one-way wave equations [4], [5], outgoing wave annihilators [6], the Liao theory [7], and the Higdon method [8].

A recent development in ABC's was Berenger's perfectly matched layer (PML) [9], which employed a fictitious, directionally dependent pair of electric and magnetic conductivities for the purpose of absorbing outgoing waves and minimizing the reflection back into the problem space.

This letter suggests a slight modification to the original Berenger implementation that is different in two ways: first, it utilizes a fictitious conductivity associated with the electric displacement, instead of the electric field. The motivation for using displacement conductivity is that those conductivities associated with the PML are completely separate from any electrical conductivity associated with the problem space. Secondly, it suggests varying the FDTD parameters directly as the most efficient means of implementing the PML, as opposed to varying the conductivities or the size of the cells, and then converting it to FDTD parameters.

II. Formulation

Starting with a transformation similar to one first proposed by Taflove and Brodwin [2]

$$\tilde{E} = \sqrt{\frac{\epsilon_0}{\mu_0}} E \tag{1a}$$

$$\tilde{D} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} D \tag{1b}$$

the Maxwell's equations can now be written as

$$\frac{\partial \tilde{\boldsymbol{D}}}{\partial t} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \cdot \nabla \times \boldsymbol{H} \tag{2a}$$

$$\tilde{\boldsymbol{D}}(\omega) = \epsilon_r^*(\omega) \cdot \tilde{\boldsymbol{E}}(\omega) \tag{2b}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\frac{1}{\sqrt{\epsilon_0 \mu_0}} \nabla \times \tilde{\boldsymbol{E}}. \tag{2c}$$

The type of formulation in (2) is used because dispersive, or even nonlinear, properties of the material are all contained in (2b) (assuming a nonmagnetic material). Numerous approaches have been taken to solve E from D in (2b), including the use of Z transforms [10]. However, that is not germane to the present discussion. What is important is that the dielectric properties of the material being simulated are expressed in (2b), regardless of how complex the material is. It will not impact the implementation of the PML in (2a) and (2c).

Of the six differential equations represented within (2a) and (2c), we will start with only the D and H components in the z direction

$$\frac{\tilde{D}_z}{\partial t} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \left(\frac{H_y}{\partial x} - \frac{H_x}{\partial y} \right) \tag{3a}$$

$$D_z = \epsilon_r^*(\omega) \cdot E_z \tag{3b}$$

$$\frac{H_z}{\partial t} = -\frac{1}{\sqrt{\epsilon_0 \mu_0}} \left(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \right). \tag{3c}$$

Berenger's approach was to introduce fictitious electric and magnetic directional conductivites and split the E and H fields in each direction into sub components, depending on the direction of the spatial derivative used to calculate it. In this letter, a slight deviation from the Berenger method will be made by introducing fictitious conductivites associated with H and D, instead of H and E

$$\frac{\tilde{D}_{zi}}{\partial t} + \frac{\sigma_{Di}}{\epsilon_0} \tilde{D}_{zi} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \left(\frac{H_{yi}}{\partial x} + \frac{H_{yk}}{\partial x} \right) \tag{4a}$$

$$\frac{\tilde{D}_{zj}}{\partial t} + \frac{\sigma_{Dj}}{\epsilon_0} \tilde{D}_{zj} = -\frac{1}{\sqrt{\epsilon_0 \mu_0}} \left(\frac{H_{xj}}{\partial y} + \frac{H_{xk}}{\partial y} \right) \tag{4b}$$

$$\tilde{D}_{zx} + \tilde{D}_{zx} = \epsilon_r^*(\omega) \cdot \tilde{E}_z \tag{4c}$$

$$\frac{H_{zi}}{\partial t} + \frac{\sigma_{Hi}}{\mu_0} H_x = -\frac{1}{\sqrt{\epsilon_0 \mu_0}} \frac{\partial \tilde{E}_y}{\partial x} \tag{4d}$$

$$\frac{H_{zj}}{\partial t} + \frac{\sigma_{Hj}}{\mu_0} H_y = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \frac{\partial \tilde{E}_x}{\partial y}. \tag{4e}$$

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The author is with the Department of Electrical Engineering, University of Idaho, Idaho Falls, ID 83402 USA.

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This is in addition to any "real" conductivity associated with the complex portion of $\epsilon_r^*(\omega)$. Start by putting (4a) in an FDTD formulation, shown in (5) at the bottom of the page. The subscript "i" in $\tilde{D}_{zi}$ refers to that portion of $\tilde{D}_z$ calculated by the spatial derivative in the x direction. Similarly, the "i" and "k" refer to the parts of H_y that had been calculated by spatial derivatives of E in the x and z direction.

The time step T is chosen as [2]

$$T = dx/(2 \cdot c_0)$$

then (4a) and (4d) become

$$\begin{aligned} &\tilde{D}_{zi}^{n+1}(i,j,k+1/2) \\ &\quad = gi(i) \cdot \tilde{D}_{zi}^{n}(i,j,k+1/2) \\ &\qquad + 0.5 \cdot (H_{yi}^{n+1/2}(i+1/2,j,k+1/2) \\ &\qquad - H_{yi}^{n+1/2}(i-1/2,j,k+1/2) \\ &\qquad + H_{yk}^{n+1/2}(i+1/2,j,k+1/2) \\ &\qquad - H_{yk}^{n+1/2}(i-1/2,j,k+1/2) \end{aligned} \tag{6a}$$

$$\begin{aligned} &H_{zi}^{n+1/2}(i+1/2,j+1/2,k) \\ &\quad = fi(i+1/2) \cdot H_{zi}^{n-1/2} \\ &\qquad \cdot (i+1/2,j+1/2,k) \\ &\qquad - 0.5 \cdot (\tilde{E}_y^n(i+1,j+1/2,k) \\ &\qquad - \tilde{E}_y^n(i,j+1/2,k)). \end{aligned} \tag{6b}$$

The PML is formed from the parameters

$$gi(i) = \left(1 - \frac{T \cdot \sigma_{Di}(i)}{\varepsilon_0}\right) \tag{7a}$$

$$fi(i+1/2) = \left(1 - \frac{T \cdot \sigma_{Hi}(i+1/2)}{\mu_o}\right). \tag{7b}$$

In the Berenger formulation, values of σ_D and σ_H are increased as one goes further into the PML to give greater absorption. Therefore, the terms $(1 - T \cdot \sigma_{Di}/\varepsilon_0)$ and $(1 - T \cdot \sigma_{Hi}/\varepsilon_0)$ will become increasingly smaller. A "common sense" stability criterion dictates that these terms cannot go below zero. Therefore, gi, gj, fi, and fj go from one at the edge of the PML to zero at the outer boundary of the FDTD space. For instance, if a PML of n layers were being developed for the X direction at the lower end (Fig. 1), the values of gi and fi would be computed as

$$gi(i) = 1 - \left(\frac{n-i}{n}\right)^3 \qquad i = 0, 1, \cdots, n \tag{8a}$$

$$fi(i+1/2) = 1 - \left(\frac{n-i-.5}{n}\right)^3 \qquad i = 0, 1, \cdots, n-1. \tag{8b}$$

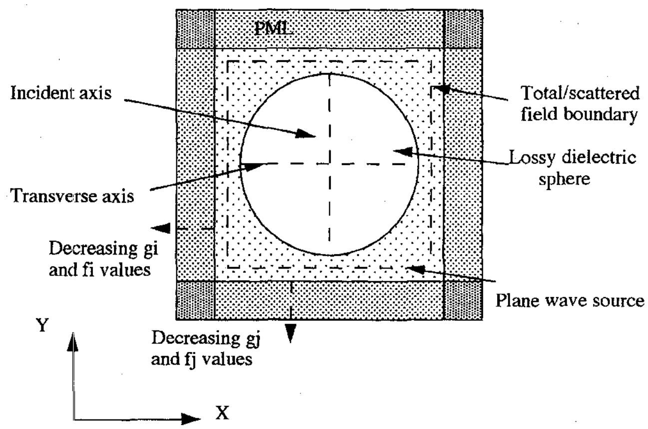

Fig. 1. Diagram of the 3-D dielectric sphere problem. The PML layer is four cells. The dielectric sphere is 20 cells in diameter. The total/scattered field boundary is five cells from the edge.

The values at the high end would be

$$gi(i_{\max} - i) = 1 - \left(\frac{n-i}{n}\right)^3 \qquad i = 0, 1, \cdots, n \tag{9a}$$

$$fi(i_{\max} - i + 1/2) = 1 - \left(\frac{n-i-.5}{n}\right)^3 \qquad i = 0, 1, \cdots, n-1 \tag{9b}$$

where $i_{\max}$ represent the dimension of the array in the x direction and n is the number of PML cells being used.

The fi values are interleaved among the gi values because the H values are interleaved among the D values in the FDTD method. The values for gj and fj and gk and fk are computed similarly.

In order to verify the accuracy of the 3-D PML, the problem illustrated in Fig. 1 was used. The problem space of $30 \times 30 \times 30$ is divided into the total and scattered fields. The cells are 1-cm cubes. A plane wave is generated at one end and subtracted out the other side. Therefore, the PML must only absorb the scattered field. The plane wave is a short gaussian pulse (0.2 ns), and the resulting frequency response of the E field in the sphere is calculated at several frequencies using a running Fourier transform. The results are then compared with a Bessel function expansion solution. (This technique has been used before in [10].) A comparison of the FDTD and the Bessel function solution along the incident and transverse axes is given in Fig. 2. As can be seen, agreement is excellent.

$$\begin{aligned} &\tilde{D}_{zi}^{n+1}(i,j,k+1/2) - \tilde{D}_{zi}^{n}(i,j,k+1/2) + \frac{T \cdot \sigma_{Di}(i)}{\epsilon_0}\tilde{D}_{zi}^{n}(i,j,k+1/2) \\ &= \left(\frac{T}{\sqrt{\epsilon_0 \mu_0}}\right) \cdot \left[\frac{H_{yi}^{n+1/2}(i+1/2,j,k+1/2) - H_{yi}^{n+1/2}(i-1/2,j,k+1/2)}{dx} + \frac{H_{yk}^{n+1/2}(i+1/2,j,k+1/2) - H_{yk}^{n+1/2}(i-1/2,j,k+1/2)}{dx}\right]. \end{aligned} \tag{5}$$

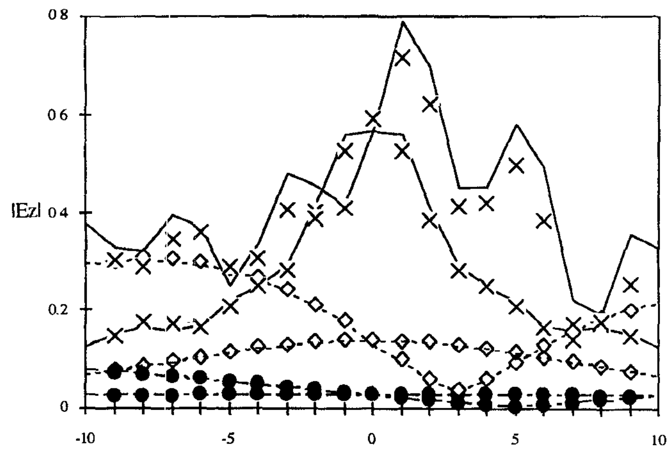

Fig. 2. Comparison of FDTD versus Bessel function expansion in calculating the resulting E field distribution in a lossy dielectric sphere illuminated by a plane wave. The Yee cells were 1-cm cubed. The sphere had a diameter = 20 cm., relative dielectric constant = 30, conductivity = .3 S/m. 700 time steps were used. By using an impulse response and a running Fourier transform, information at three frequencies 50, 200, and 800 MHz were obtained with one run. The symbols represent the FDTD values and the lines the Bessel values.

III. Comparison with Berenger's PML

Because the PML described in this letter is implemented throughout the problem space, it is decidedly computationally more expensive than Berenger's method, both in computer memory requirements and in CPU time requirements.

A. Core Memory

1) Twice as many D field and H field matrices are needed.
2) The parameters gi, gj, gk, fi, fj, fk are additionally needed. However, these are one-dimensional matrices.

For the same number of Yee cells in a 3-D problem, this results in a 12% increase in core memory requirement.

B. CPU Time

There is also an additional computational requirement:

1) Because each D field is split, the computation needed for the calculation of the D field is doubled.
2) The H field is also split, but each computation is the derivative of the E field in one direction instead of two. Therefore, twice as many difference calculations are needed, but each is less computationally intense.

This PML will add to the computational requirements slightly because the computations of the D and H fields are split. It results in a 20% increase in CPU time.

In contrast, Berenger's PML has little additional memory and time requirements because it is only implemented around the boundaries.

IV. Discussion

This simplified method has the following advantages:

1) No special treatment is needed in the PML as opposed to the rest of the FDTD problem space. The values of $gi, gj, fi,$ and fj are just one in the FDTD problem space. Therefore, dispersive or even nonlinear materials can be modeled by (3b).
2) It is easier to formulate the PML because the values of $gi, gj, fi,$ and fj are computed directly. It is not necessary to determine corresponding fictitious conductivities.

As pointed out in the previous section, this PML is computationally less efficient than Berenger's. In this method, the emphasis is place on ease of use, particularly in lossy or dispersive background media. Whether or not this justifies the increase in computer resources is for the user to decide.

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